Laplace and Fourier Transforms
- Definitions and Properties

Adam Szmigielski
aszmigie@pjwstk.edu.pl
room s07 (robotics lab)
/public/aszmigie/ELK

The Laplace transform

- is a useful analytical tool for converting time-domain *(t)* signal descriptions into functions of a complex variable *(s)*. This complex domain description of a signal provides new insight into the analysis of signals and systems.
- In addition, the Laplace transform method often simplifies the calculations involved in obtaining system response signals.

Definition

The Laplace transform of the continuous-time signal it x (t) is

$$X(s) = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} d(t) \tag{1}$$

The variable *s* that appears in this integrand exponential is generally complex valued and is therefore often expressed in terms of its rectangular coordinates

$$s = \sigma + j\omega, \tag{2}$$

where $\sigma = Re(s)$ and $\omega = Im(s)$ are referred to as the real and imaginary components of s, respectively.

The signal $x(t)$ and its associated Laplace transform $X(s)$ are said to form a Laplace *transform pair*. This reflects a form of equivalency between the two apparently different entities $x(t)$ and $X(s)$. We may symbolize this interrelationship in the following suggestive manner:

$$X(s) = \pounds[x(t)] \tag{3}$$

where the operator notation $\pounds$ means to multiply the signal $x(t)$ being operated upon by the complex exponential e^{-st} and then to integrate that product over the time interval $(-\infty, +\infty)$.

	Time Signal $x(t)$	Laplace Transform $X(s)$
1.	$e^{-at}u(t)$	$\frac{1}{s+a}$
2.	$t^k e^{-at}u(-t)$	$\frac{k!}{(s+a)^{k+1}}$
3.	$-e^{-at}u(-t)$	$\frac{1}{(s+a)}$
4.	$(-t)^k e^{-at}u(-t)$	$\frac{k!}{(s+a)^{k+1}}$
5.	$u(t)$	$\frac{1}{s}$
6.	$\delta(t)$	1
7.	$\frac{d^k\delta(t)}{dt^k}$	s^k
8.	$t^k u(t)$	$\frac{k!}{s^{k+1}}$
9.	$\operatorname{sgn} t = \begin{cases} 1, t \geq 0 \\ -1, t < 0 \end{cases}$	$\frac{2}{s}$
10.	$\sin \omega_0 t\ u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$
11.	$\cos \omega_0 t\ u(t)$	$\frac{s}{s^2+\omega_0^2}$
12.	$e^{-at} \sin \omega_0 t\ u(t)$	$\frac{\omega}{(s+a)^2+\omega_0^2}$
13.	$e^{-at} \cos \omega_0 t\ u(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$

Properties of Laplace Transform

- Let us obtain the Laplace transform of a signal, $x(t)$, that is composed of a linear combination of two other signals,

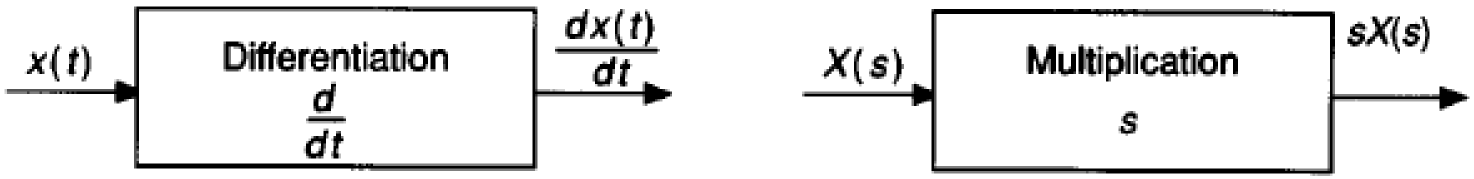
$$x(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t) \tag{4}$$

where α_1 and a_2 are constants.

- The **linearity property** indicates that:

$$\pounds[\alpha_1 x_1(t)+\alpha_2 x_2(t)] = \alpha_1 \pounds[x_1(t)]+\alpha_2 \pounds[x_2(t)] = \alpha X_1(s)+\alpha_2 X2(s) \tag{5}$$

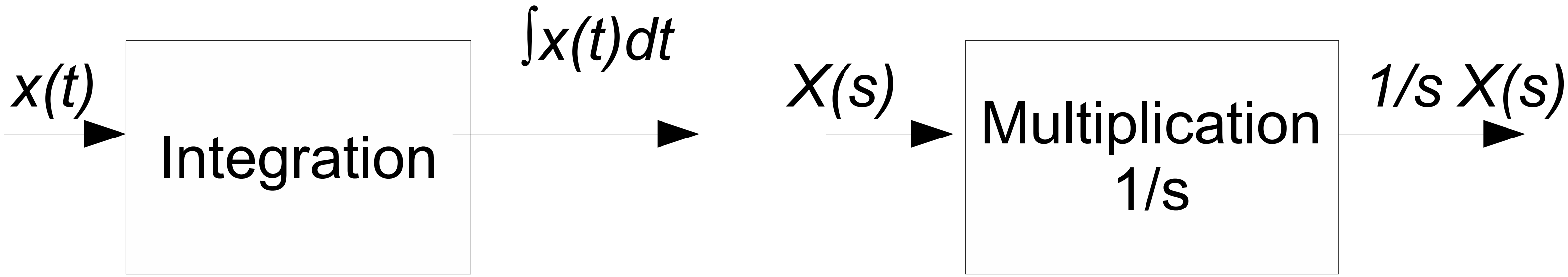
Time-Domain Differentiation



The operation of time-domain differentiation has then been found to correspond to a multiplication by s in the Laplace variable s domain. The Laplace transform of differentiated signal $\frac{dx(t)}{dt}$ is

$$\pounds[\frac{dx(t)}{dt}] = s \cdot X(s) \tag{6}$$

Time-Domain Integration



The operation of time-domain Integration has then been found to correspond to a multiplication by $\frac{1}{s}$ in the Laplace variable s domain. The Laplace transform of integration signal $\int x(t)dt$ is

$$\pounds[\int x(t)dt] = \frac{1}{s} \cdot X(s) \tag{7}$$

Time Shift

The signal $x(t - t_0)$ is said to be a version of the signal $x(t)$ right shifted (or delayed) by $t0$ seconds. Right shifting (delaying) a signal by a $t0$ second duration in the time domain is seen to correspond to a multiplication by e^{-st_0} in the Laplace transform domain. The desired Laplace transform relationship is

$$\pounds[x(t - t_0)] = X(s) \cdot e^{-st_0} \tag{8}$$

where $X(s)$ denotes the Laplace transform of the unshifted signal $x(t)$.

Time-Convolution Property

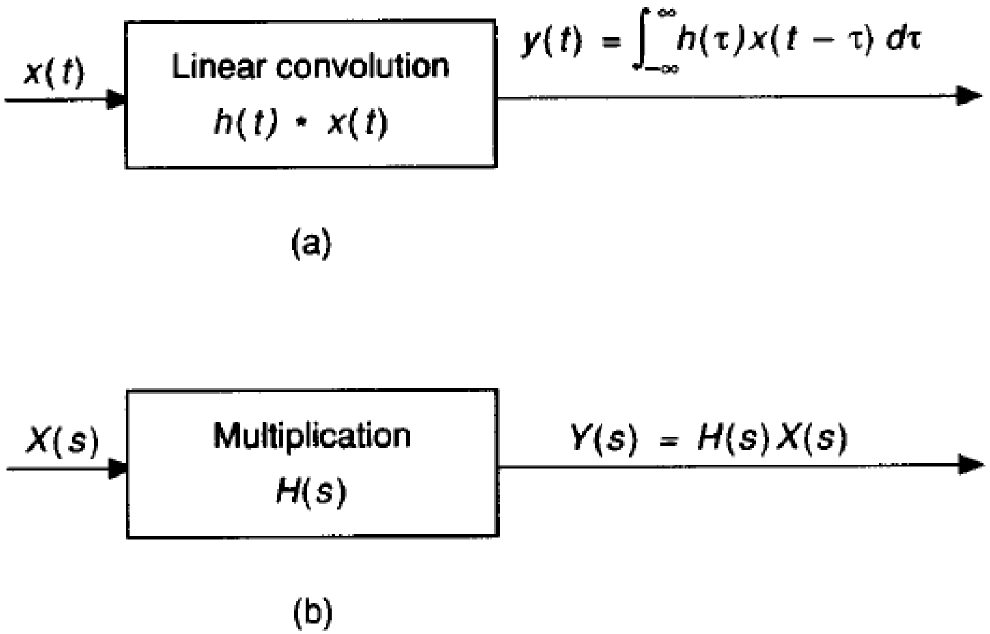
The convolution integral signal $y(t)$ can be expressed as

$$y(t) = \int_{-\infty}^{\infty} h(\tau) \cdot x(t-\tau) d\tau \tag{9}$$

where $x(t)$ denotes the input signal, the $h(t)$ characteristic signal identifying the operation process. The Laplace transform of the response signal is simply given by

$$Y(s) = H(s) \cdot X(s) \tag{10}$$

where $H(s) = \pounds[h(t)]$ and $X(s) = \pounds[x(t)]$. Thus, the convolution of two time-domain signals is seen to correspond to the multiplication of their respective Laplace transforms in the $s - domain$



Function $H(s)$ is also called **transfer function** and from eq. (10) can be expressed as

$$H(s) = \frac{Y(s)}{X(s)} \tag{11}$$

Inverse Laplace Transform

Given a transform function $X(s)$ and its region of convergence, the procedure for finding the signal $x(t)$ that generated that transform is called finding the inverse Laplace transform and is symbolically denoted as

$$x(t) = \pounds^{-1}[X(s)] \tag{12}$$

The signal $x(t)$ can be recovered by means of the relationship

$$x(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} X(s) \cdot e^{st} ds \tag{13}$$

Fourier transform - Frequency-Domain

- Laplace transform - s-domain

$$X(s) = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} d(t) \tag{14}$$

- Fourier transform - Frequency-Domain (by replacing s with $j\omega$)

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} d(t) \tag{15}$$

where $w = 2\pi f$, f - frequency, $j = \sqrt{-1}$ - imaginary unit.

The Transformed Circuit

Instead of writing the describing circuit equations, transforming the results, and solving for the transform of the circuit current or voltage, we may go directly to a transformed circuit, which is the original circuit with the currents, voltages, sources, and passive elements replaced by transformed equivalents. The current or voltage transforms are then found using ordinary circuit theory and the results inverted to the time-domain answers.

Inductor Transformation

For an inductance L, the voltage is

$$v_L(t) = L \cdot \frac{di_L(t)}{dt} \tag{16}$$

Transforming, we have

$$V_L(s) = L \cdot s \cdot I_L(s) \tag{17}$$

Ratio $\frac{V(s)}{I(s)}$ is an **impedance**. For inductor impedance is equal $X_l(s) = s \cdot L$ (In frequency domain $X_l(j\omega) = j\omega \cdot L$)

Capacitor Transformation

In the case of a capacitance C we have

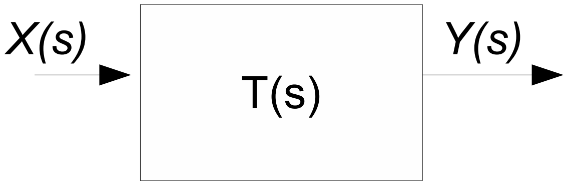
$$v_C(t) = \frac{1}{C}\int i_C(t)dt \tag{18}$$

which transforms to

$$V_C(s) = \frac{1}{C}\cdot\frac{1}{s}I_C(s) \tag{19}$$

For capacitor impedance is equal $X_c(s) = \frac{1}{s \cdot C}$ (In frequency domain $X_c(j\omega) = \frac{1}{j\omega \cdot C}$)

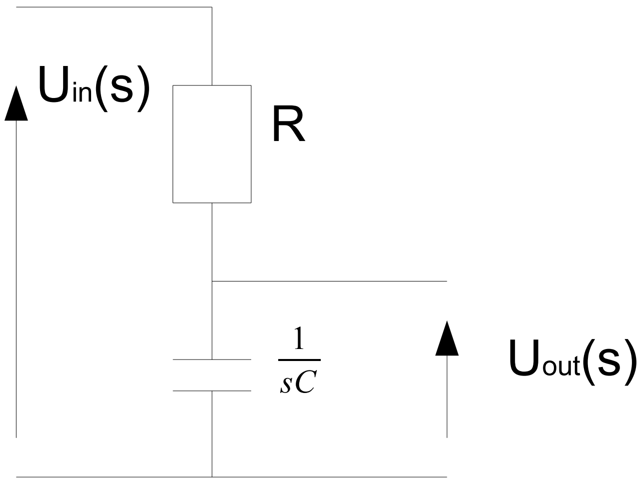
Transfer functions (*transmittance*)



$$T(s) = \frac{Y(s)}{H(s)}$$

A transfer function is a mathematical representation (in terms of frequency), of the relation between the input and output of a (linear time-invariant) system

(*Transmittance*) - example

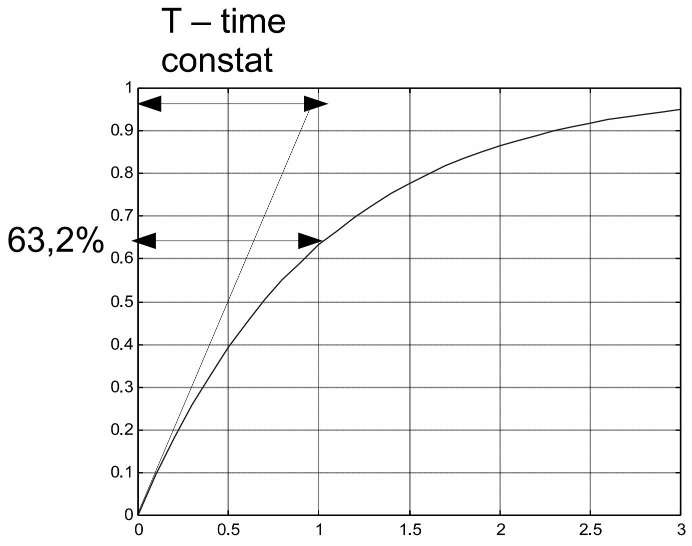


Transfer function can be calculated as follow:

$$T(s) = \frac{U_{out}(s)}{U_{in}(s)} = \frac{\frac{1}{sC}}{R + \frac{1}{sC}} = \frac{1}{1 + s \cdot RC} = \frac{1}{1 + s \cdot T} \qquad (20)$$

where $T = R \cdot C$ is called **time constant**

Time constant



In a capacitor-resistor circuit, the number of seconds required for the capacitor to reach 63.2% of its full charge after a voltage is applied. The time constant of a capacitor with a capacitance (C) in farads in series with a resistance (R) in ohms is equal to $R \cdot C$ seconds.

Magnitude and phase diagrams

For transfer function $T(s) = \frac{1}{1+sT}$ $(T = 5)$