TWIERDZENIE O TRZECH CIĄGACH

Ciągi $\{a_n\}, \{b_n\}, \{c_n\}$ takie, że: $a_n \le b_n \le c_n$ i jeśli $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = g$, to $\lim_{n\to\infty} b_n = g$

$$\lim_{n\to\infty} \sqrt[n]{a} = 1 \text{ (przy } a > 0)$$

Przykład:

A)

$$\lim_{n\to\infty} \sqrt[n]{7 * 5^n} = \lim_{n\to\infty} \sqrt[n]{7}\ (\to \mathbf{1}) * \sqrt[n]{5^n}\ (\to \mathbf{5}) = 5$$

B)

$$\lim_{n\to\infty} \sqrt[n]{3 * 10^n + 7 * 8^n} = \mathbf{10}$$

$$\sqrt[n]{10^n}\ (\to \mathbf{10}) \le \sqrt[n]{3 * 10^n + 7 * 8^n}\ (\to \mathbf{10}) \le \sqrt[n]{3 * 10^n + 7 * 10^n} = \sqrt[n]{10 * 10^n}\ (\to \mathbf{10})$$

C) zadanie analogiczne do tych z kolokwiów

$$\lim_{n\to\infty} \sqrt[n]{\frac{2 * 5^n + 3 * 2^n + 157}{4 * 3^n + 2 * 2^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{2 * 5^n + 3 * 2^n + 157}}{\sqrt[n]{4 * 3^n + 2 * 2^n}} = \frac{\mathbf{5}}{\mathbf{3}}$$

$$\sqrt[n]{5^n}\ (\to \mathbf{5}) \le \sqrt[n]{2 * 5^n + 3 * 2^n + 157}\ (\to \mathbf{5}) \le \sqrt[n]{2 * 5^n + 3 * 5^n + 5^n} = \sqrt[n]{6 * 5^n}$$

$$\sqrt[n]{3^n} \le \sqrt[n]{4 * 3^n + 2 * 2^n}\ (\to \mathbf{3}) \le \sqrt[n]{6 * 3^n} (\to \mathbf{3})$$

D)

$$\lim_{n\to\infty} \frac{2n^2 + \sin(n!)}{4n^2 - 3\cos(n^2)} = \frac{1}{2}$$

$$\begin{cases} -1 \le \sin(n!) \le 1 \\ -1 \le \cos(n^2) \le 1 \end{cases}$$

$$\frac{\mathbf{2n^2 - 1}}{\mathbf{4n^2 + 3}} \le \frac{2n^2 + \sin(n!)}{4n^2 - 3\cos(n^2)} \le \frac{2n^2 + 1}{4n^2 - 3} \left(\to \frac{\mathbf{1}}{\mathbf{2}}\right)$$

$$\frac{2 - \frac{1}{n^2}\ (\to 0)}{4 + \frac{3}{n^2}\ (\to 0)} \to \frac{\mathbf{1}}{\mathbf{2}}$$

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n \quad ; \quad e = \lim_{a_n\to\infty} \left(1 + \frac{1}{a_n}\right)^{a_n}$$

$$e \approx 2{,}71 \quad ; \quad a^{n*m} = (a^n)^m$$

E)

$$\lim_{n\to\infty} \left(\frac{n+2}{n+5}\right)^{2n} = \lim_{n\to\infty} \left(\frac{n+5-3}{n+5}\right)^{2n} = \lim_{n\to\infty} \left(1 + \frac{-3}{n+5}\right)^{2n * \frac{n+5}{-3} * \frac{-3}{n+5}} =$$

$$= \lim_{n\to\infty} \left[\left(1 + \frac{-3}{n+5}\right)^{\frac{n+5}{-3}}\ (\to e)\right]^{\frac{-6n}{n+5}} = e^{\lim_{n\to\infty}\left(\frac{-6n}{n+5}\right)} = e^{-6}$$

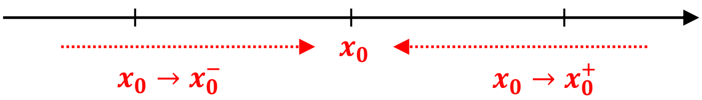
F)

$$\lim_{n\to\infty}\left(\frac{2n-1}{2n+5}\right)^{3n}=\lim_{n\to\infty}\left(\frac{2n+5-6}{2n+5}\right)^{3n}=\lim_{n\to\infty}\left(1+\frac{6}{2n+5}\right)^{3n}=$$

$$=\lim_{n\to\infty}\left(1+\frac{-6}{2n+5}\right)^{\frac{2n+5}{-6}*\frac{-6}{2n+5}*3n}=\lim_{n\to\infty}\left[\left(1+\frac{-6}{2n+5}\right)^{\frac{2n+5}{-6}}\ (\to e)\right]^{-\frac{18n}{2n+5}}=e^{-9}$$

GRANICA FUNKCJI

$$\lim_{x\to x_0} f(x)$$



Granica funkcji wg Cauchy'ego:

$$\lim_{x\to x_0} f(x)=g$$

$$\forall_{\epsilon>0}\ \exists_{\sigma>0}\ |x-x_0|<g \Longrightarrow |f(x)-g|<\epsilon$$

Granica funkcji wg Heinego:

$$\forall_{x_n}(x_n\to x_0)\Longrightarrow f(x_n)=g$$

Granica funkcji istnieje, jeśli: $\lim\limits_{x\to x_0^-} f(x)=\lim\limits_{x\to x_0^+} f(x)$

<u>Przykład</u>:

A)

$$\lim_{x\to x_0}\frac{1}{x}$$

$$\lim_{x\to 0^+}\frac{1}{x}=\left[\frac{1}{0^+}\right]=+\infty$$

$$\lim_{x\to 0^-}\frac{1}{x}=\left[\frac{1}{0^-}\right]=-\infty$$

<u>Odp.: Granica nie istnieje</u>

B)

$$\lim_{x\to x_0}\frac{x}{|x|}\qquad |x|=\begin{cases}x \text{ dla } x\geq 0\\ -x \text{ dla } x<0\end{cases}$$

$$\lim_{x\to 0^+}\frac{x}{|x|}=\lim_{x\to 0^+}\frac{x}{x}=1$$

$$\lim_{x\to 0^-}\frac{x}{|x|}=\lim_{x\to 0^-}\frac{x}{-x}=-1$$

<u>Odp.: Granica nie istnieje</u>

<u>Zadanie</u>:

Oblicz granicę funkcji

A)

$$\lim_{x\to 1}\frac{x^2+2x+1}{x^3+1}\quad \therefore \textbf{Najpierw podstawiamy wartość}$$

$$\lim_{x\to 1}\frac{x^2+2x+1}{x^3+1}=\frac{4}{2}=\mathbf{2}$$

B)

$$\lim_{x\to1}\frac{x^2-2x+1}{x^3-1}=\lim_{x\to1}\frac{(x-1)^{2}}{(x-1)(x^2+x+1)}=\left[\frac{0}{3}\right]=\mathbf{0}$$

C)

$$\lim_{x\to5}\frac{x-5}{x^2-3x-10}\lim_{x\to5}\frac{x-5}{(x-5)(x+2)}=\frac{\mathbf{1}}{\mathbf{7}}$$

$$x^2-3x-10=0$$

$$\Delta=3^2-4*1*(-10)=49$$

$$x_{1,2}=\frac{3\pm\sqrt{49}}{2}=5\,,(-2)$$

D)

$$\lim_{x\to\infty}\frac{x^2-4x+7}{4x^2+9x+150}\quad\therefore\ \textbf{Dzielimy przez najwyższą potęgę mianownika}$$

$$\lim_{x\to\infty}\frac{1-\frac{4}{x}\,(\to\mathbf{0})+\frac{7}{x^2}\,(\to\mathbf{0})}{4+\frac{9}{x}\,(\to\mathbf{0})+\frac{150}{x^2}\,(\to\mathbf{0})}=\frac{\mathbf{1}}{\mathbf{4}}$$

E)

$$\lim_{x\to0^+}\frac{2x^5+3x^3-6x}{x^5+x^2}\quad\therefore\ \textbf{Dzielimy przez najniższą potęgę mianownika}$$

$$\lim_{x\to0^+}\frac{2x^3\,(\to\mathbf{0})+3x\,(\to\mathbf{0})-\frac{6}{x}\,(\to-\infty)}{x^3\,(\to\mathbf{0})+1}=-\infty$$

F) przykład z zeszłorocznego kolokwium

$$\lim_{x\to\infty}\left(\sqrt{x^2+2x+1}-\sqrt{x^2+3}\right)$$

$$\lim_{x\to\infty}\left(\sqrt{x^2+2x+1}-\sqrt{x^2+3}\right)*\frac{\sqrt{x^2+2x+1}+\sqrt{x^2+3}}{\sqrt{x^2+2x+1}+\sqrt{x^2+3}}=\lim_{x\to\infty}\frac{x^2+2x+1-x^2-3}{\sqrt{x^2+2x+1}+\sqrt{x^2+3}}=$$

$$=\lim_{x\to\infty}\frac{2x-2}{\sqrt{x^2+2x+1}+\sqrt{x^2+3}}=\lim_{x\to\infty}\frac{2-\frac{2}{x}\,(\to\mathbf{0})}{\sqrt{1+\frac{2}{x}\,(\to\mathbf{0})+\frac{1}{x^2}\,(\to\mathbf{0})}+\sqrt{1+\frac{3}{x^2}\,(\to\mathbf{0})}}=\frac{2}{2}=\mathbf{1}$$

G)

$$\lim_{x\to3^-}2^{\frac{1}{x-3}}=[2^{-\infty}]=\left[\frac{1}{2^\infty}\right]=\mathbf{0}$$

$$\lim_{x\to3^-}\frac{1}{x-3}=\left[\frac{1}{0^-}\right]=-\infty$$

$$a^{-n}=\frac{1}{a^n}$$

H)

$$\lim_{x\to0}\frac{\left(\sqrt{x^2+1}-\sqrt{x+1}\right)}{\left(1-\sqrt{x+1}\right)}$$

mnożymy

$$\lim_{x\to0}\frac{\left(\sqrt{x^2+1}-\sqrt{x+1}\right)}{\left(1-\sqrt{x+1}\right)}*\frac{1+\sqrt{x+1}}{\left(1+\sqrt{x+1}\right)}*\frac{\left(\sqrt{x^2+1}+\sqrt{x+1}\right)}{\sqrt{x^2+1}+\sqrt{x+1}}$$

mnożymy

$$\lim_{x\to 0}\frac{(x^2+\cancel{1}-x-\cancel{1})(1+\sqrt{x+1})}{(\cancel{1}-x-\cancel{1})(\sqrt{x^2+1}+\sqrt{x+1})}=\lim_{x\to 0}\frac{\cancel{x}(x-1)(1+\sqrt{x+1})}{-\cancel{x}(\sqrt{x^2+1}+\sqrt{x+1})}=\lim_{x\to 0}\frac{(x-1)(1+\sqrt{x+1})}{(\sqrt{x^2+1}+\sqrt{x+1})}$$

I)

$$\lim_{x\to\infty}\left(\frac{3x^2-1}{3x^2+5}\right)^{3x^3+4}=\lim_{x\to\infty}\left(\frac{3x^2+5-6}{3x^2+5}\right)^{3x^3+4}=\lim_{x\to\infty}\left(1+\frac{-6}{3x^2+5}\right)^{\frac{3x^2+5}{-6}*\frac{-6}{3x^2+5}*3x^3+4}=$$

$$=\lim_{x\to\infty}\left[\left(1+\frac{-6}{3x^2+5}\right)^{\frac{3x^2+5}{-6}}(\to e)\right]^{\frac{-18x^3-24}{3x^2+5}}=[e^{-\infty}]=\left[\frac{1}{e^{\infty}}\right]=\mathbf{0}$$

$$\lim_{x\to\infty}\frac{-18x^3-24}{3x^2+5}=\frac{-18x\,(-\infty)-\frac{24}{x^2}\,(\to 0)}{3+\frac{5}{x^2}\,(\to 0)}=-\infty$$

J)

$$\lim_{x\to 0}\sqrt[2x]{1+10x}=\lim_{x\to 0}(1+10x)^{\frac{1}{2x}}$$

$$\frac{1}{x}\to\infty\ ;\ \frac{1}{x}=t\Longrightarrow x=\frac{1}{t}$$

$$\lim_{t\to 0^+}\left(1+\frac{10}{t}\right)^{\frac{t}{2}*\frac{5}{5}}\left(\left(1+\frac{10}{t}\right)^{\frac{t}{2}*\frac{1}{5}}\to e\right)=e^5$$

$$t=-\frac{1}{x}\Longrightarrow x=-\frac{1}{t}\quad;\quad t\to+\infty$$

$$\lim_{t\to 0^-}\left(1-\frac{10}{t}\right)^{-\frac{t}{2}*\frac{5}{5}}\left(\left(1-\frac{10}{t}\right)^{-\frac{t}{2}*\frac{1}{5}}\to e\right)=e^5$$

Jeżeli x dąży do 0, to liczymy $x\to 0^+$ i $x\to 0^-$.

CIĄGŁOŚĆ FUNKCJI

Funkcja jest ciągła w x_0 wtedy, gdy:

1) istnieje wartość w tym punkcie,
2) istnieje granica $\lim_{x\to x_0} f(x)$; $\left[\lim_{x\to x_0^-} f(x)=\lim_{x\to x_0^+} f(x)\right]$
3) $f(x_0)=\lim_{x\to x_0} f(x)$

__Zadanie__:
Zbadać, czy funkcja jest ciągła:

$$f(x)=\begin{cases}\dfrac{\sin x}{x} & \text{dla } x\neq 0\\ 2 & \text{dla } x=0\end{cases}$$

$$f(0)=2\ \neq\ \lim_{x\to 0}\frac{\sin x}{x}=1$$

Odp.: Funkcja nie jest ciągła

Zadanie:
Zbadać dla jakich wartości a i b funkcja jest ciągła:

A)

$$f(x)=\begin{cases} x+2 \ \text{dla} \ x<0 \\ a \ \text{dla} \ x=0 \\ 3x^2-4ax+b \ \text{dla} \ x>0 \end{cases}$$

$$\begin{cases} f(0)=a \\ \lim\limits_{x\to 0^-} x+2=2 \end{cases} \Rightarrow \boldsymbol{a=2}$$

$$\begin{cases} a=2 \\ \lim\limits_{x\to 0^+} 3x^2-4ax+b=2 \end{cases} \Rightarrow \boldsymbol{b=2}$$

B)

$$f(x)=\begin{cases} x^3-1 \ \text{dla} \ x<0 \\ a+b \ \text{dla} \ x=0 \\ 3x+b \ \text{dla} \ x>0 \end{cases}$$

$$\lim_{x\to 0^-} x^3-1=-1=f(0)=a+b$$

$$\lim_{x\to 0^+} 3x+b=b=f(0)=a+b$$

$$\begin{cases} a+b=b \\ a+b=-1 \end{cases}$$

$$\begin{cases} \boldsymbol{a=0} \\ \boldsymbol{b=-1} \end{cases}$$

C)

$$f(x)=\begin{cases} ax+b \ dla \ x<3 \\ 7 \ dla \ 3\le x\le 5 \\ ax^2+b \ dla \ x>5 \end{cases}$$

$$f(3)=7$$

$$7=\lim_{x\to 3^-} ax+b=3a+b$$

$$f(5)=7$$

$$7=\lim_{x\to 5^+} ax^2+b=25a+b$$

$$\begin{cases} 3a+b=7 \quad \ddots * (-1) \\ 25a+b=7 \end{cases}$$

$$\begin{cases} -3a-b=-7 \\ 25a+b=7 \end{cases}$$

$$\begin{cases} 22a=0 \ \Rightarrow \ \boldsymbol{a=0} \\ -3*0-b=-7 \ \Rightarrow \ \boldsymbol{b=7} \end{cases}$$

D)

$$f(x)=\begin{cases} -2\sin x \ \ dla \ x\le -\frac{\pi}{2} \\ a*\sin x+b \ dla \ -\frac{\pi}{2}\le x\le \frac{\pi}{2} \\ \cos x \ \ dla \ x\ge \frac{\pi}{2} \end{cases}$$

$$f\left(-\frac{\pi}{2}\right)=-2\sin\left(-\frac{\pi}{2}\right)=2$$

$$2=\lim_{x\to\left(-\frac{\pi}{2}\right)^+} a*\sin x+b=-a+b$$

$$f\left(\frac{\pi}{2}\right)=\cos\left(\frac{\pi}{2}\right)=0$$

$$0 = \lim_{x\to\left(\frac{\pi}{2}\right)^-} a * \sin x + b = a + b$$

$$\begin{cases} -a + b = 2 \\ a + b = 0 \end{cases}$$

$$\begin{cases} 2b = 2 \implies \boldsymbol{b = 1} \\ \qquad \boldsymbol{a = -1} \end{cases}$$

Granice ciągów podwójnych:

$$\lim_{(x,y)\to(x_0\ ,y_0)} f(x,y) = g \Leftrightarrow \forall_{x_n,y_n}(x_n - x_0\ \wedge y_n \to y_0) \to f(x_n\ ,y_n) = g$$

Przykłady ciągów zbieżnych do 0:

$$\frac{1}{n}, \frac{1}{2^n}, \frac{2}{n}, -\frac{1}{n}, \frac{1}{\sqrt{n}}, \frac{1}{n^5}, \ldots.$$

Przykłady ciągów zbieżnych do 1:

$$n^0, \sqrt[n]{n}, 1 + \frac{1}{n}, 1 - \frac{1}{n}, 1 - \frac{2}{n}, \ldots.$$

Przykład:

Wykazać, że nie istnieje poniższa granica.

$$\lim_{(x_0\ ,y_0)\to(0\ ,0)} \frac{x}{y}$$

$$x_n = \frac{1}{n}\ ;\ y_n = \frac{1}{n} \quad \to \quad \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n}} = \mathbf{1}$$

$$x_n = \frac{2}{n}\ ;\ y_n = \frac{1}{n} \quad \to \quad \lim_{n\to\infty} \frac{\frac{2}{n}}{\frac{1}{n}} = 2$$

Odp.: Granica nie istnieje

Zadanie:

Wykazać, czy istnieje granica funkcji

A)

$$\lim_{(x,y)\to(0,0)} \frac{x}{x+y}$$

$$x_n = \frac{1}{n}\ ;\ y_n = \frac{1}{n} \quad \to \quad \lim_{n\to 0} \frac{\frac{1}{n}}{\frac{1}{n}+\frac{1}{n}} = \frac{\frac{1}{n}}{\frac{2}{n}} = \mathbf{\frac{1}{2}}$$

$$x_n = \frac{3}{n}\ ;\ y = \frac{4}{n} \quad \to \quad \lim_{n\to 0} \frac{\frac{3}{n}}{\frac{3}{n}+\frac{4}{n}} = \mathbf{\frac{3}{7}}$$

Odp.: Granica nie istnieje

B)

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x+y}$$

$$x_n = \frac{1}{n}\ ;\ y_n = \frac{1}{n} \quad \to \quad \lim_{n\to 0} \frac{\frac{1}{n^2}}{\frac{2}{n}} = \frac{1}{2n} = \mathbf{0}$$

$$x_n = \frac{1}{n}\ ;\ y_n = \frac{1}{n^2} \quad \to \quad \lim_{n\to 0} \frac{\frac{1}{n^3}}{\frac{1}{n}+\frac{1}{n^2}} = \frac{\frac{1}{n^2}}{1+\frac{1}{n}} = \mathbf{0}$$

$$x_n = \frac{1}{n} \;;\; y_n = -\frac{1}{n} \;\rightarrow\; \lim_{n\to\infty} \frac{-\frac{1}{n^2}}{0} = -\infty$$

<u>Odp.: Granica nie istnieje</u>

C)

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$$

$$x_n = \frac{1}{n} \;;\; y_n = \frac{1}{n} \;\rightarrow\; \lim_{n\to\infty} \frac{\frac{1}{n^3}}{\frac{1}{n^2}+\frac{1}{n^4}} = \frac{\frac{1}{n^3}}{\frac{1}{n^2}\left(1+\frac{1}{n^2}\right)} = \frac{\frac{1}{n}}{1+\frac{1}{n^2}} = \mathbf{0}$$

$$x_n = \frac{1}{n} \;;\; y_n = \frac{1}{\sqrt{n}} \;\rightarrow\; \lim_{n\to\infty} \frac{\frac{1}{n} * \frac{1}{n}}{\frac{1}{n^2}+\frac{1}{n^2}} = \frac{\frac{1}{n^2}}{\frac{2}{n^2}} = \mathbf{\frac{1}{2}}$$

<u>Odp.: Granica nie istnieje</u>

Granice sterowane

$$A = \lim_{x\to x_0}\left(\lim_{y\to y_0} f(x,y)\right) \quad ; \quad B = \lim_{y\to y_0}\left(\lim_{x\to x_0} f(x,y)\right)$$

Jeśli $A \neq B$ to granica nie istnieje.

<u>**Zadanie**</u>:
Uzasadnij, że nie istnieje granica poniższych funkcji:

A)

$$\lim_{(x,y)\to(0,0)} \frac{5x-3y+xy}{3x-5y+x^2}$$

$$\lim_{x\to 0}\left(\lim_{y\to 0} \frac{5x-3y+xy}{3x-5y+x^2}\right) = \lim_{x\to 0} \frac{5x}{3x+x^2} = \mathbf{\frac{5}{3}}$$

$$\lim_{y\to 0}\left(\lim_{x\to 0} \frac{5x-3y+xy}{3x-5y+x^2}\right) = \lim_{y\to 0} \frac{-3y}{-5y} = \mathbf{\frac{3}{5}}$$

<u>Odp.: Granica nie istnieje</u>

B)

$$\lim_{(x,y)\to(0,0)} \frac{5xy-3y+x}{3xy+y-2x}$$

$$\lim_{x\to 0}\left(\lim_{y\to 0} \frac{5xy-3y+x}{3xy+y-2x}\right) = \lim_{x\to 0} \frac{x}{-2x} = -\mathbf{\frac{1}{2}}$$

$$\lim_{y\to 0}\left(\lim_{x\to 0} \frac{5xy-3y+x}{3xy+y-2x}\right) = \lim_{y\to 0} -\frac{3y}{y} = \mathbf{-3}$$

<u>Odp.: Granica nie istnieje</u>

C)

$$\lim_{(x,y)\to(1,-2)} \frac{(x-1)^3+y+2}{(x-1)^2-y-2}$$

$$\lim_{x\to 1}\left(\lim_{y\to -2} \frac{(x-1)^3+y+2}{(x-1)^2-y-2}\right) = \lim_{x\to 1} \frac{(x-1)^{\cancel{3}}}{\cancel{(x-1)^2}} = \mathbf{0}$$

$$\lim_{y\to -2}\left(\lim_{x\to 1} \frac{(x-1)^3+y+2}{(x-1)^2-y-2}\right) = \lim_{y\to -2} \frac{y+2}{-y-2} = \lim_{y\to -2} \frac{\cancel{y+2}}{-1\cancel{(y+2)}} = \mathbf{-1}$$

<u>Odp.: Granica nie istnieje</u>