DC Linear Circuit Analysis

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Kirchhoff's Current Law KCL *(Kirchhoff's first law)*

At any point in an electrical circuit the sum of currents flowing towards that point is equal to the sum of currents flowing away from that point.

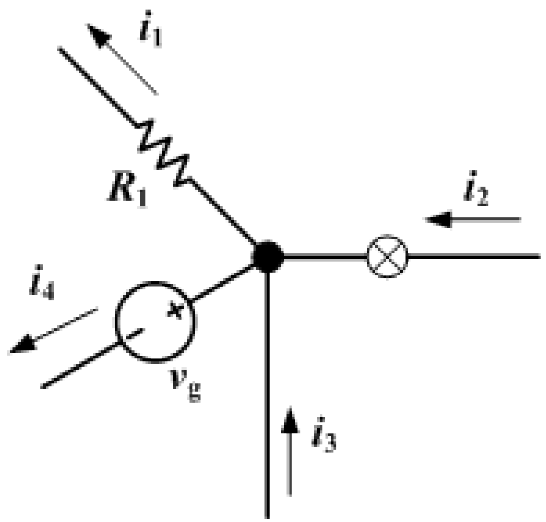


Figure 1: The current entering any junction is equal to the current leaving that junction. $i_1(t) + i_4(t) = i_2(t) + i_3$(t)

Kirchhoff's Current Law can be stated alternatively as:

"the algebraic sum of the branch currents entering (or leaving) any node of a circuit at any instant of time must be zero."

In this form, the label of any current whose orientation is away from the node is preceded by a minus sign. The currents entering node must satisfy

$$i_2(t) + i_3(t) - i_1(t) - i_4(t) = 0$$

In general, the currents entering or leaving each node of a circuit must satisfy. Sign "+" says that current flows into the node and "-" flows out.

$$\sum_k i_k(t) = 0 \tag{1}$$

Kirchhoff's Voltage Law KVL *(Kirchhoff's second law)*

The directed sum of the electrical potential differences around a closed circuit must be zero.

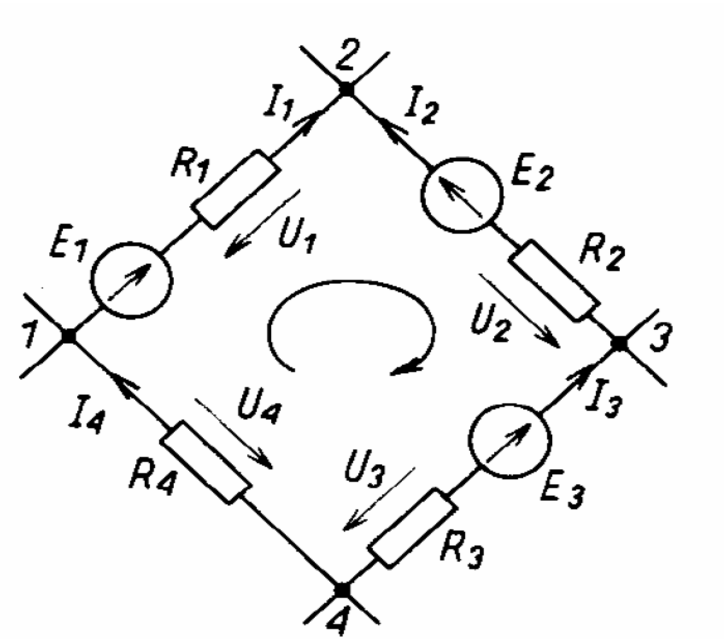


Figure 2: The sum of all the voltages around the loop is equal to zero.
$e_1(t) - v_1(t) - e_2(t) + v_2(t) - e_3(t) + v_3(t) - v_4(t) = 0$

Kirchhoff’s Voltage Law

KVL can be expressed mathematically as “the algebraic sum of the voltages drops around any closed path of a circuit at any instant of time is zero.” This statement can also be cast as an equation:

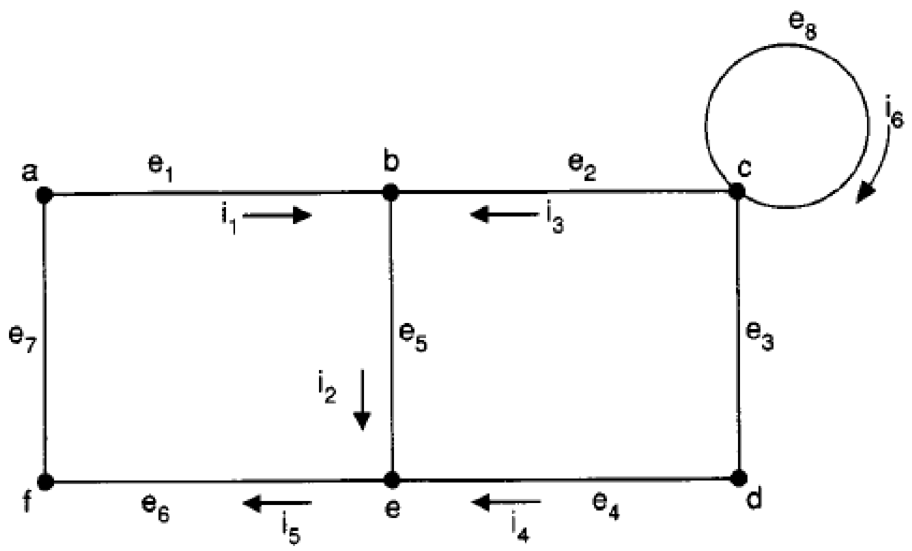
$$\sum_k V_k(t) = 0 \tag{2}$$

Ohm’s law

The resistance of a resistor can be defined in terms of the voltage drop across the resistor and current through the resistor related by Ohm’s law

$$R = \frac{U}{I} \tag{3}$$

where R is the resistance $[\Omega]$, U is the voltage across the resistor $[V]$, and I is the current through the resistor $[A]$. Whenever a current is passed through a resistor, a voltage is dropped across the ends of the resistor.



- A circuit has a topological (or graph) view (consisting of a labeled set of nodes and a labeled set of edges.)
- Each edge is associated with a pair of nodes (a node is drawn as a dot and represents a connection between two or more physical components)
- An edge is drawn as a line and represents a path, or branch, for current flow through a component
- Each current has a designated direction, usually denoted by an arrow symbol.

Given a branch, the pair of nodes to which the branch is attached defines the convention for measuring voltages in the circuit. Given the ordered pair of nodes (a, b), a voltage measurement is formed as follows:

$$v_{ab} = v_a - v_b \quad (4)$$

where v_a and v_b are the absolute electrical potentials (voltages) at the respective nodes, taken relative to some reference node. The measured quantity, v_{ab} , is called the *voltage drop* from node a to node b. We note that

$v_{ab} = -v_{ba}$ and that $v_{ba} = v_b - v_a$

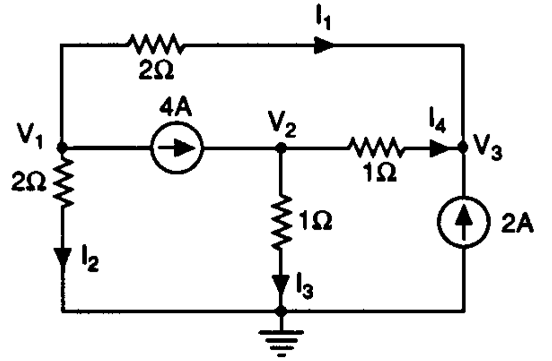
is called the voltage rise from a to b. Each node voltage implicitly defines the voltage drop between the respective node and the ground node.

Node Analysis

In a node analysis, the node voltages are the variables in a circuit, and KCL is the vehicle used to determine them. One node in the network is selected as a *reference node*, and then all other node voltages are defined with respect to that particular node. This reference node is typically referred to as ground using the symbol ($\perp$), indicating that it is at *ground-zero potential.*

Therefore, as a general rule:
If the node voltages (potentials) are known, all branch currents in the network can be immediately determined.
In order to determine the node voltages in a network, we apply KCL to every node in the network except the reference node. Therefore, given an N-node circuit, we employ N–1 linearly independent equations.

Node Analysis - Example



For 4 nodes we write 4 -1 equations:

$$\begin{cases} I_1 + 4 + I_2 = 0 \\ -4 + I_3 + I_4 = 0 \\ -I_1 - I_4 - 2 = 0 \end{cases} \tag{5}$$

Using Ohm's law these equations can be expressed as

$$\begin{cases} \frac{V_1 - V_3}{2} + 4 + \frac{V_1}{2} = 0 \\ -4 + \frac{V_2}{1} + \frac{V_2 - V_3}{1} = 0 \\ -\frac{V_1 - V_3}{2} - \frac{V_2 - V_3}{1} - 2 = 0 \end{cases} \tag{6}$$

and solved as follow:

Node potentials:

$$\begin{cases} V_1 = -\frac{8}{3} \\ V_2 = \frac{10}{3} \\ V_3 = \frac{8}{3} \end{cases} \tag{7}$$

and currents:

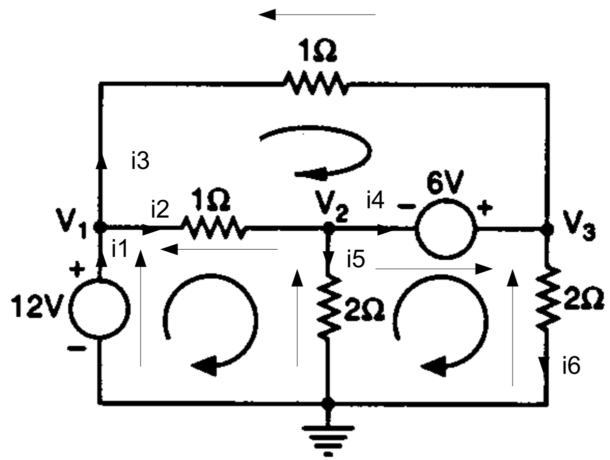
$$\begin{cases} I_1 = \frac{V_1 - V_3}{2} = -\frac{8}{3} \\ I_2 = \frac{V_1}{2} = -\frac{8}{6} \\ I_3 = \frac{V_2}{1} = \frac{10}{3} \\ I_4 = \frac{V_2 - V_3}{1} = \frac{2}{3} \end{cases} \tag{8}$$

If current is negative (I_1 and I_2) the direction of current flow must be changed.

Mesh Analysis - algorithm

1. mark with vectors all currents and voltages
2. For each mesh write KVL
3. for $n-1$ nodes write KCL
4. solve equations

Mesh Analysis - an example



For 3 nodes we can write equations:

$$
\begin{aligned}
I_1 &= I_2 + I_3 \\
I_2 &= I_4 + I_5 \\
I_6 &= I_3 + I_4
\end{aligned} \tag{9}
$$

For each mesh (3 meshes) we write equation:

$$
\begin{aligned}
12 - 1 \cdot I_2 - 2 \cdot I_5 = 0 \\
2 \cdot I_5 + 6 - 2 \cdot I_6 = 0 \\
1 \cdot I_2 - 1 \cdot I_3 - 6 = 0
\end{aligned} \tag{10}
$$

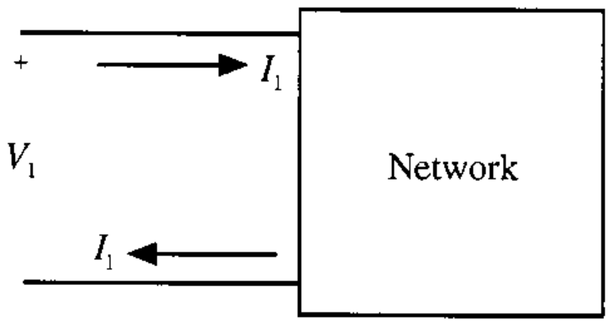
Finally we have 6 equations with 6 variable:

$$\begin{cases} I_1 = I_2 + I_3 \\ I_2 = I_4 + I_5 \\ I_6 = I_3 + I_4 \\ 12 - 1 \cdot I_2 - 2 \cdot I_5 = 0 \\ 2 \cdot I_5 + 6 - 2 \cdot I_6 = 0 \\ 1 \cdot I_2 - 1 \cdot I_3 - 6 = 0 \end{cases} \tag{11}$$

with solution

$$\begin{cases} I_1 = 8 \\ I_2 = 7 \\ I_3 = 1 \\ I_4 = 4.5 \\ I_5 = 2.5 \\ I_6 = 5.5 \end{cases} \tag{12}$$

One-Port

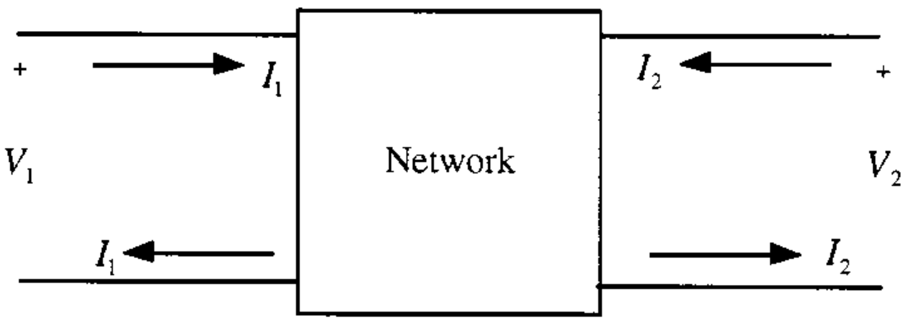


Many times we want to model the behavior of an electric network at only two terminals as shown in Figure. Here, only V_1 and I_1, not voltages and currents internal to the circuit, need to be described. We define the pair of terminals shown as a port, where the current, I_1, entering one terminal equals the current leaving the other terminal.

We can mathematically model the network at the port as

$$V_1 = Z \cdot I_1$$

Two-Port Networks



We can model such circuits as two-port networks as shown in Figure. Here we see the input port, represented by V_1 and I_1, and the output port, represented by V_2 and I_2. Currents are assumed positive if they flow as shown in Figure.

Mathematical Modeling of Two-Port Networks via z Parameters (Impedance Parameters)

$$\begin{cases} V_1 = z_{11}I_1 + z_{12}I_2 \\ V_2 = z_{21}I_1 + z_{22}I_2 \end{cases} \tag{13}$$

or as matrix equation

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \cdot \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \tag{14}$$

Other two-port descriptions

- Admittance Parameters

$$\begin{cases} I_1 = y_{11}V_1 + y_{12}V_2 \\ I_2 = y_{21}V_1 + y_{22}V_2 \end{cases} \tag{15}$$

- Hybrid Parameters

$$\begin{cases} V_1 = h_{11}I_1 + h_{12}V_2 \\ I_2 = h_{21}I_1 + h_{22}V_2 \end{cases} \tag{16}$$

- others ...

Thevenin’s and Norton’s equivalents

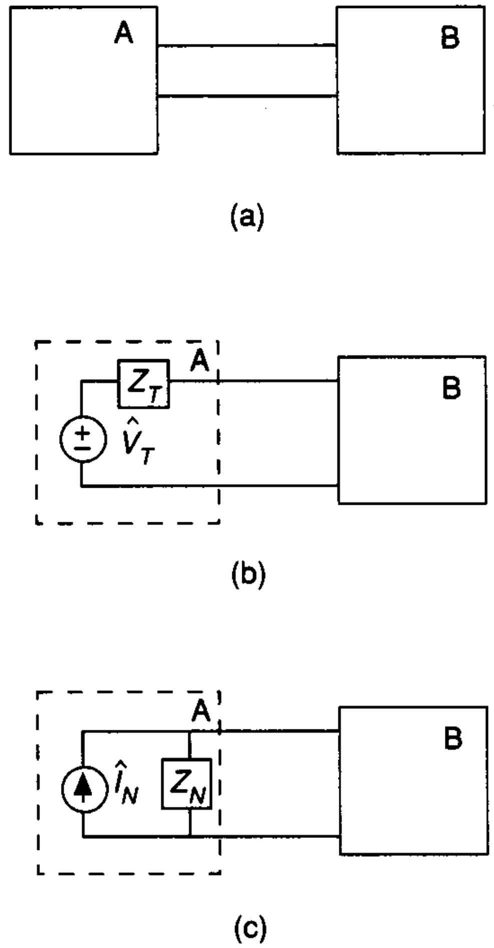


Figure 3:

(a) Two one-port networks

(b) the Thévenin equivalent for network a

(c) the Norton equivalent for network a.

Thevenin's Theorems

The statement of the Thevenin theorem is based on Fig. 3(b)

Insofar as a load which has no magnetic or controlled source coupling to a one-port is concerned, a network containing linear elements and both independent and controlled sources may be replaced by an ideal voltage source of strength, $\widehat{V_T}$ and an equivalent impedance Z_T , in series with the source. The value of $\widehat{V_T}$ is the open-circuit voltage, $\widehat{V_{OC}}$, appearing across the terminals of the network and Z_T is the driving point impedance at the terminals of the network, obtained with all independent sources set equal to zero.

Norton's Theorems

The statement of the Norton theorem is based on Fig. 3(c)

Insofar as a load which has no magnetic or controlled source coupling to a one-port is concerned, the network containing linear elements and both independent and controlled sources may be replaced by an ideal current source of strength, $\widehat{I_N}$, and an equivalent impedance, Z_N, in parallel with the source. The value of $\widehat{I_N}$ is the short-circuit current, $\widehat{I_{SC}}$, which results when the terminals of the network are shorted and Z_N is the driving point impedance at the terminals when all independent sources are set equal to zero.

The Equivalent Impedance, $Z_T = Z_N$

The first method involves the direct calculation of $Z_T = Z_N$ by looking into the terminals of the network after all independent sources have been nulled. Independent sources are nulled in a network by replacing all independent voltage sources with a short circuit and all independent current sources with an open circuit.

Linearity and superposition

If cause and effect are linearly related, the total effect due to several causes acting simultaneously is equal to the sum of the individual effects due to each of the causes acting one at a time.

Linearity and superposition

$$\widehat{I} = \widehat{I_C} + \widehat{I_V}$$

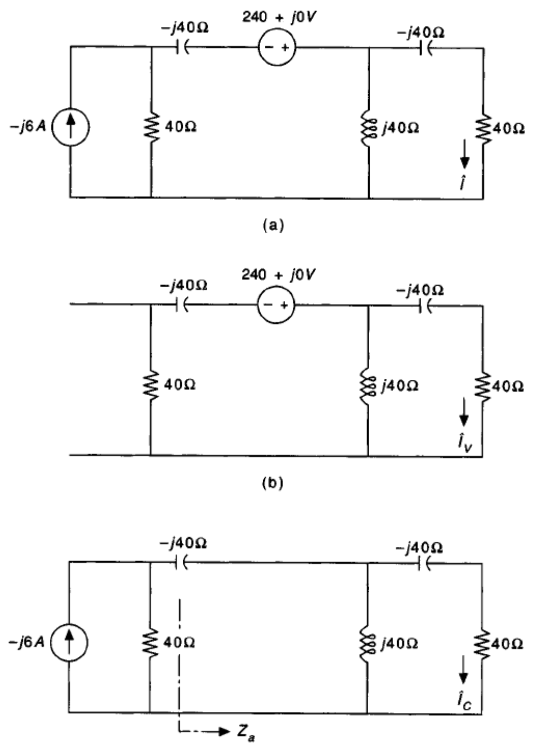


Figure 4:

(a) A network to be solved by using superposition

(b) the network with the current source nulled (current source is replaced with open circuit)

(c) the network with the voltage source nulled. (voltage source is replaced with short circuit)

Tellegen’s Theorem

In an arbitrarily lumped network subject to KVL and KCL constraints, with reference directions of the branch currents and branch voltages associated with the KVL and KCL constraints, the product of all branch currents and branch voltages must equal zero.

Tellegen’s theorem may be summarized by the equation

$$\sum_{k=1}^{b} v_k \cdot j_k = 0 \tag{17}$$

where the lower case letters v and j represent instantaneous values of the branch voltages and branch currents, respectively, and where b is the total number of branches.

Maximum Power Transfer

$$Z_{in} = Z_{out}$$