Laplace and Fourier Transforms
- Applications

Adam Szmigielski
aszmigie@pjwstk.edu.pl
room s07 (robotics lab)
/public/aszmigie/ELK

Repetition

- Laplace Transform is a useful analytical tool for converting time-domain *(t)* signal descriptions into unctions of a complex variable *(s)*. This complex domain description of a signal provides new insight into the analysis of signals and systems.
- The Laplace transform often simplifies the calculations of differential and integral equation (integral "becomes" multiplying by $\frac{1}{s}$ and differential by s
- In addition, the Laplace transform method often simplifies the calculations involved in obtaining system response signals.

Differentiation Theorems

If we replace $f(t)$ in the one-sided transform by its derivative $f'(t)$ and integrate by parts, we have the transform of the derivative

$$\pounds[f'(t)] = s \cdot F(s) - f(0)$$

We may formally replace f by f' to obtain

$$\pounds[f''(t)] = s \cdot \pounds[f'(t)] - f'(0)$$

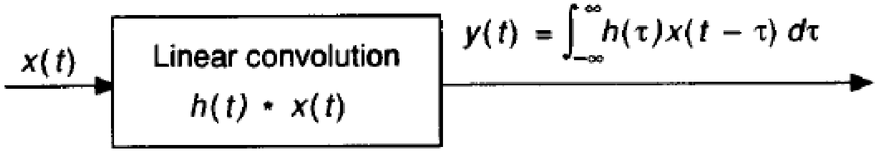
or by $\pounds[f''(t)] = s^2 \cdot F(s) - s \cdot f(0) - f'(0)$
Generally:

$$\pounds[f^n(t)] = s^n \cdot F(s) - s^{n-1} \cdot f(0) - s^{n-2} \cdot f'(0) \ldots - f^{n-1}(0)$$

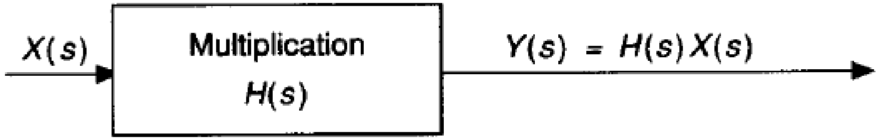
The signal $x(t)$ and its associated Laplace transform $X(s)$ are said to form a Laplace *transform pair*. This reflects a form of equivalency between the two apparently different entities $x(t)$ and $X(s)$. We may symbolize this interrelationship in the following suggestive manner:

$$X(s) = \pounds[x(t)] \tag{1}$$

Transfer function, transmittance



(a)



(b)

Function $H(s)$ is also called **transfer function** or *transmittance* and can be expressed as

$$H(s) = \frac{Y(s)}{X(s)} \tag{2}$$

Ohm's law

The resistance of a resistor can be defined in terms of the voltage drop across the resistor and current through the resistor related by Ohm's law

$$R = \frac{U}{I} \tag{3}$$

where R is the resistance $[\Omega]$, V is the voltage across the resistor $[V]$, and I is the current through the resistor $[A]$. Whenever a current is passed through a resistor, a voltage is dropped across the ends of the resistor.

Impedance = Resistance + j*Reactance

- For alternating signals electrical resistance can be describe by complex number and denoted as impedance Z

$$Z = R + j \cdot X \tag{4}$$

where real part of Z is an resistance R and imaginary part of Z $j \cdot X$ is called reactance. Reactance represents "resistance of real capacitor" or "resistance of real inductor" and depends of frequency.

- Impedance can be expressed in the notion of Ohm's law :

$$Z = \frac{U}{I} \tag{5}$$

Ideal Capacitor Reactance

In the case of a capacitance C we have $v_C(t) = \frac{1}{C}\int i_C(t)dt$ which transforms to

$$V_C(s) = \frac{1}{C} \cdot \frac{1}{s} I_C(s) \tag{6}$$

For capacitor impedance (for ideal capacitor reactance=impedance) is equal

$$X_c(s) = \frac{V_C(s)}{I_C(s)} = \frac{1}{s \cdot C} \tag{7}$$

In frequency domain

$$X_c(j\omega) = \frac{1}{j\omega \cdot C} \tag{8}$$

Ideal Inductor Reactance

For an inductance L, the voltage is $v_L(t) = L \cdot \frac{di_L(t)}{dt}$. Transforming, we have

$$V_L(s) = L \cdot s \cdot I_L(s) \tag{9}$$

For inductor impedance (for ideal inductor reactance=impedance) is equal

$$X_l(s) = \frac{V_L}{I_L} = s \cdot L \tag{10}$$

In frequency domain

$$X_l(j\omega) = j\omega \cdot L \tag{11}$$

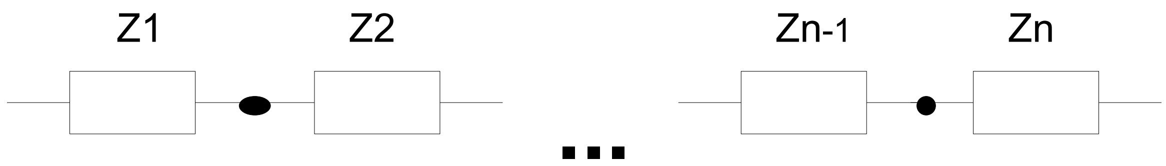
Properties of capacitor and inductor reactance

- observe, that for capacitor:
  $X_C = \frac{1}{j\omega \cdot C} = -j\frac{1}{\omega \cdot C}$ hence $X_C < 0$
- similar for inductor:
  $X_L = j\omega \cdot L$ hence $X_L > 0$
- If *reactance* is negative circuit has an *capacitive character*
- If *reactance* is positive circuit has an *inductive character*

How capacitor and inductor reactance depends on frequency ?

$\omega = 0$	$X_C = \frac{1}{j\omega \cdot C} = \infty$	$X_L = j\omega \cdot L = 0$
$\omega \uparrow$	$X_C \downarrow$	$X_L \uparrow$
$\omega \downarrow$	$X_C \uparrow$	$X_L \downarrow$
$\omega = \infty$	$X_C = \frac{1}{j\omega \cdot C} = 0$	$X_L = j\omega \cdot L = \infty$

Impedance Networks - impedances connected in series

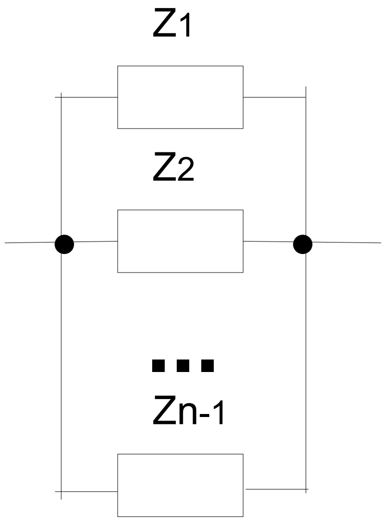


$$Z = Z_1 + Z_2 + \ldots Z_{n-1} + Z_n = \sum_{i=1}^{n} Z_i \qquad (12)$$

Special cases

- resistance: $R = R_1 + R_2 + \ldots R_{n-1} + R_n$
- capacity: $\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \ldots + \frac{1}{C_{n-1}} + \frac{1}{Cn}$
- inductance: $L = L_1 + L_2 + \ldots L_{n-1} + L_n$

Impedance Networks - impedances connected in parallel.

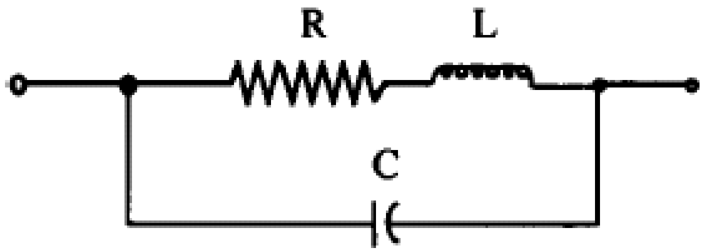


$$\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2} + \ldots + \frac{1}{Z_n} = \sum_{i=1}^{n} \frac{1}{Z_i} \tag{13}$$

Special cases

- resistance: $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \ldots + \frac{1}{Rn}$
- capacity: $C = C_1 + C_2 + \ldots C_{n-1} + C_n$
- inductance: $\frac{1}{L} = \frac{1}{L_1} + \frac{1}{L_2} + \ldots + \frac{1}{Ln}$

Impedance Networks - other example

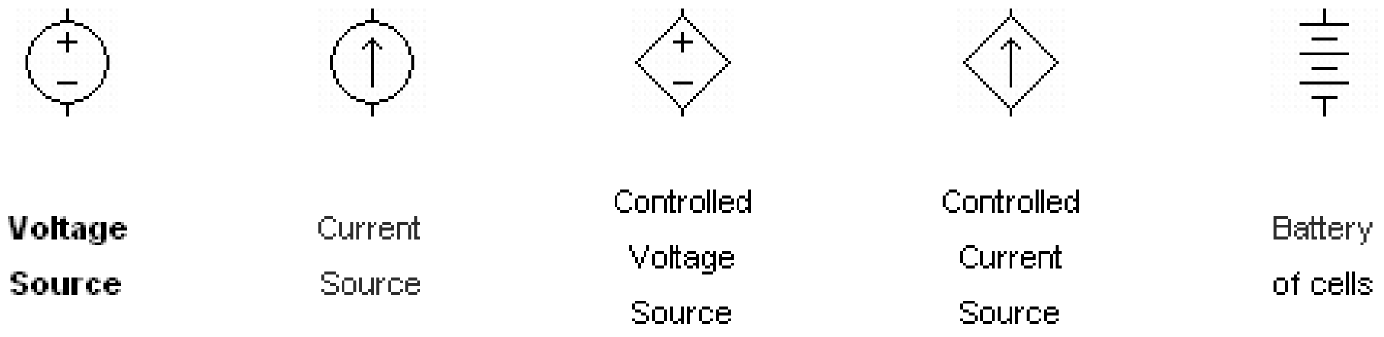


Equivalent circuit for a resistor.

$$\frac{1}{Z} = \frac{1}{sC} + \frac{1}{R + sL} \tag{14}$$

$$Z = \frac{\frac{1}{sC} \cdot (R + sL)}{\frac{1}{sC} + R + sL} = \frac{R + sL}{1 + sRC + s^2 LC} \tag{15}$$

Voltage and current sources



Properties ideal sources

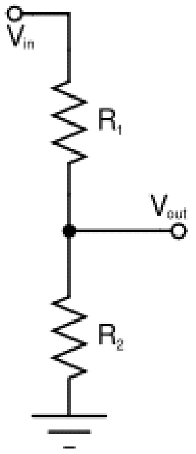
Current source:

-

Voltage source:

-

Voltage divider

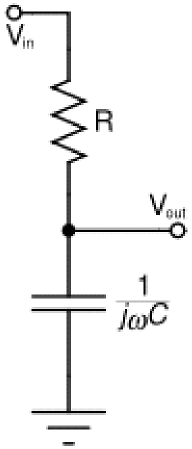


$$V_{\text{out}} = \frac{R_2}{R_1 + R_2} \cdot V_{\text{in}}$$

Voltage divider is a simple device designed to create a voltage (V_{out}) which is proportional to another voltage (V_{in}). It is commonly used to create a reference voltage, and may also be used as a signal attenuator at low frequencies. Voltage dividers are also known by the terms *resistor divider*.

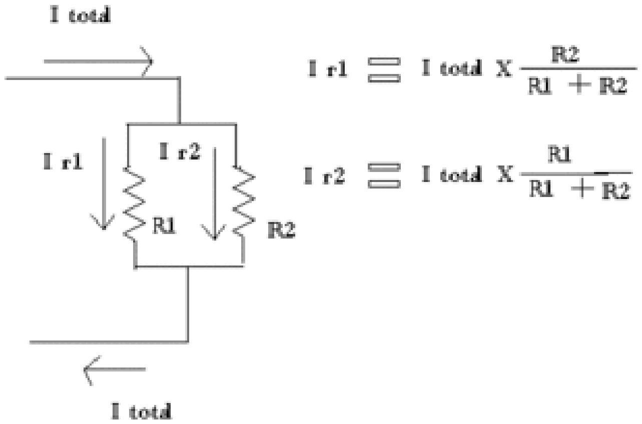
Impedance divider

$$V_{\text{out}} = \frac{Z_2}{Z_1 + Z_2} \cdot V_{\text{in}}$$



A voltage divider is usually thought of as two resistors, but for electronics signals at a given frequency capacitors, inductors, or any combined impedance can be used.
The ratio contains an imaginary number, and actually contains both the amplitude and phase shift information of the filter. To extract just the amplitude ratio, calculate the magnitude of the ratio, or just use the reactance of the capacitor instead of the impedance.

Current divider



If two or more impedances are in parallel to each other, the current that enters them will be split between them in inverse proportion to their resistance (from Ohm's law). It also follows that if the impedances have the same value the current is split equally.

This is a general form of the current divider.

$$I_x = \frac{R_t}{R_x} \cdot I_y \tag{16}$$

Thevenin's and Norton's Theorems

Thevenin’s and Norton’s Theorems