$$\lim_{(x,y)\to(0,0)} \frac{\sin(4xy)}{\sin(7xy)} * \frac{7xy * 4}{4xy * 7} = \frac{4}{7}$$

$$\frac{\sin(4xy)}{4xy} \to 1 \ ; \ \frac{7xy}{\sin(7xy)} \to 1$$

$$\lim_{(x,y)\to 0} \frac{\sin(x^2+y^2)}{2(x^2+y^2)} = \frac{1}{2}$$

$$\lim_{(x,y)\to(0,0)} \frac{4-\sqrt{xy+16}}{xy} * \frac{4+\sqrt{xy+16}}{4+\sqrt{xy+16}} = \lim_{(x,y)\to(0,0)} \frac{16-xy-16}{xy*\left(4+\sqrt{xy+16}\right)} = -\frac{1}{8} \ \ (xy \to 0)$$

$$\lim_{(x,y)\to(0,3)} \frac{3x}{tg4xy} * \frac{4y}{4y} = \frac{3}{12} = \frac{1}{4}$$

POCHODNE

Pochodna jest to miara przyrostu funkcji:

$$f'(x) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$$

Pochodna mówi nam jak szybko zmienia się funkcja.

Przykład:
Obliczyć z definicji :

$$f(x) = \frac{1}{x^2}$$

$$f'(x) = \lim_{x\to x_0} \frac{\frac{1}{x^2}-\frac{1}{x_0^2}}{x-x_0} = \lim_{x\to x_0} \frac{\frac{x_0^2-x^2}{x^2*x_0^2}}{x-x_0} = \lim_{x\to x_0} \frac{\cancel{(x_0-x)}(x_0+x)}{x^2*x_0^2*\cancel{(x_0-x)}} = \frac{-1(x_0+x_0)}{x_0^2*x_0^2} = -\frac{2x_0}{x_0^4} = -\frac{2}{x_0^3}$$

$$(a*f') = a*f'$$

$$(fg)' = f'*g+f*g'$$

$$\left(\frac{f}{g}\right)' = \frac{f'*g-f*g'}{g^2}$$

$$\left(f(g(x))\right)' = f'(g(x))*g'(x)$$

Zadanie:
Obliczyć pochodne funkcji:

A)

$$f(x) = 2x^5-3x^3+27x-1$$

$$f'(x) = 2*5x^4-3*3x^2+27*1-0$$

B)

$$f(x) = 7\sqrt[3]{x^2}-5\sqrt{x^5}+\frac{2}{\sqrt[5]{x^2}}+\frac{1}{x} = 7*x^{\frac{2}{3}}-5x^{\frac{5}{2}}+2x^{-\frac{2}{5}}+x^{-1}$$

$$f'(x) = 7*\frac{2}{3}x^{-\frac{1}{3}}-5*\frac{5}{2}x^{\frac{3}{2}}+2*\left(-\frac{2}{5}x^{-\frac{7}{5}}\right)+(-x^{-2})$$

C)

$$f(x) = \frac{3}{2x-5} = 3 * (2x-5)^{-1}$$
$$f'(x) = 3 * (-1)(2x-5)^{-2} * 2$$

D)

$$f(x) = \sin(7x)$$
$$f'(x) = \cos(7x) * 7$$

E)

$$f(x) = (7x^2 - \cos(3x))^5$$
$$f'(x) = 5 * (7x^2 - \cos(3x))^4 * (7 * 2x - (-\sin(3x)) * 3\ )$$

F)

$$f(y) = \sqrt{\frac{1-y}{3+2y^2}}$$
$$f'(y) = \frac{1}{2\sqrt{\frac{1-y}{3+2y^2}}} * \frac{-1 * (3+2y^2) - (1-y) * 4y}{(3+2y^2)^2}$$

G)

$$f(x) = cosx - \frac{1}{3}\cos^3 x$$
$$f'(x) = -sinx - \frac{1}{3} * 3\cos^2 x * (-\sin(x))$$

H)

$$f(x) = \sqrt{1 + tg\left(x + \frac{1}{x}\right)}$$
$$f'(x) = \frac{1}{2\sqrt{1 + tg(x + \frac{1}{x})}} * \frac{1}{\cos^2\left(x + \frac{1}{x}\right)} * (1 - x^{-2})$$

I)

$$f(x) = arctg\sqrt{\frac{1-x}{1+x}}$$
$$f'(x) = \frac{1}{1 + \left(\sqrt{\frac{1-x}{1+x}}\right)^2} * \frac{1}{2 * \sqrt{\frac{1-x}{1+x}}} * \frac{-1(1+x) - (1-x) * 1}{(1+x)^2}$$

J)

$$f(x) = \sin(2x) * (\ln(x^2) * e^{3x})$$
$$f'(x) = 2\cos(2x) * (\ln(x^2) * e^{3x}) + \sin(2x) * (\frac{1}{x^2} * 2x * e^{3x} + \ln x^2 * 3 * e^{3x})$$

K)

$$f(x) = 3 * e^{2\sin^3 x}$$
$$f'(x) = 3 * e^{2\sin^3 x} * 2 * 3\sin^2 x * (-\cos x)$$

L) z zeszłorocznego kolokwium

$$f(x) = \sqrt[5]{\frac{\cos(\ln x)}{ctg(x^5)}} = \left(\frac{\cos\ln(x)}{ctg(x^5)}\right)^{\frac{1}{5}}$$

$$f'(x) = \frac{1}{5} * \left(\frac{\cos\ln(x)}{ctg(x^5)}\right)^{-\frac{4}{5}} * \left(\frac{-\sin(\ln x) * \frac{1}{x} * ctg(x^5) - \cos(\ln x) * \frac{1}{\sin^2(x^5)} * 5x^4}{ctg^2(x^5)}\right)$$

M) z zeszłorocznego kolokwium

$$f(x) = \sqrt{\sin\left(\frac{x - 2^x}{tg(x^2 - 3x)}\right)}$$

$$f'(x) = \frac{1}{2\sqrt{\sin\left(\frac{x-2^x}{tg(x^2-3x)}\right)}} * \cos\left(\frac{x - 2^x}{tg(x^2 - 3x)}\right) * \\ * \frac{(1 - 2^x * \ln 2) * tg(x^2 - 3x) - (x - 2^x) * \left(\frac{1}{\cos^2(x^2 - 3x)} * (2x - 3)\right)}{tg^2(x^2 - 3x)}$$

N) z zeszłorocznego kolokwium

$$f(x) = \sin^2\left(\frac{\ln(x^5)}{arctg(2x - 1)}\right)$$

$$f'(x) = 2\sin\left(\frac{\ln(x^5)}{arctg(2x - 1)}\right) * \frac{\frac{1}{x^5} * 5x^4 * arctg(2x - 1) - \ln(x^5) - \frac{1}{1 + (2x - 1)^2 * 2}}{arctg^2(2x - 1)}$$

Pochodne funkcji logarytmicznych:

$$e^{\ln a} = a$$

$$\ln a^n = nln\ a$$

<u>**Zadanie**</u>:

Oblicz pochodną funkcji:

A)

$$f(x) = \sin x^x = e^{\ln(\sin x)^x} = e^{xln(\sin x)}$$

$$f'(x) = e^{xln(\sin x)} * \left(1 * \ln(\sin x) + x * \frac{1}{\sin x} \_ \cos x\right)$$

B)

$$f(x) = tgx^{\ln\left(\frac{1}{x}\right)} = e^{\ln\left(tgx^{\ln\left(\frac{1}{x}\right)}\right)} = e^{\ln\left(\frac{1}{x}\right) * \ln(tgx)}$$

$$f'(x) = e^{\ln\left(\frac{1}{x}\right) * \ln(tgx)} * \left(\ln\left(\frac{1}{x}\right) * \ln(tgx)\right)'$$

$$f'(x) = e^{\ln\left(\frac{1}{x}\right) * \ln(tgx)} * (x * (-x^{-2}) * \ln(tgx)) + \ln\frac{1}{x} * \frac{1}{tgx} * \frac{1}{\cos^2}$$

REGUŁA DE L'HOSPITALA

Jeśli $\lim\limits_{x\to x_0}\frac{f(x)}{g(x)}=\left[\frac{0}{0}\right]\ lub\ \left[\frac{\infty}{\infty}\right]\ to\ \lim\limits_{x\to x_0}\frac{f(x)}{g(x)}=\lim\limits_{x\to x_0}\frac{f'(x)}{g'(x)}$

$$\lim_{x\to x_0} f(x)*g(x)=[0*\infty]=\lim_{x\to x_0}\frac{f(x)}{g^{-1}(x)}$$

Przykład:

Obliczyć granicę

A)

$$\lim_{x\to 0}\frac{e^x-e^{-x}}{x}=\left[\frac{1-1}{0}\right]=\lim_{x\to 0}\frac{e^x-e^{-x}*(-1)}{1}=\lim_{x\to 0}e^x-e^{-x}=\mathbf{2}$$

B)

$$\lim_{x\to 0}\frac{\sin x-x*\cos x}{x^3}=\left[\frac{0}{0}\right]^H=\lim_{x\to 0}\frac{\cos x-\left(cosx_x*(-sinx)\right)}{3x^2}=\left[\lim_{x\to 0}\frac{xsinx}{3x^2}\right]^H=\lim_{x\to 0}\frac{cosx}{3}=\frac{\mathbf{1}}{\mathbf{3}}$$

C)

$$\lim_{x\to 0}ctgx*x=[\infty*0)=\lim_{x\to 0}\frac{x}{ctg^{-1}x}=\left[\lim_{x\to 0}\frac{x}{tgx}\right]^H=\lim_{x\to 0}\frac{1}{\frac{1}{\cos^2 x}}=\mathbf{1}$$

D)

$$\lim_{x\to 1}(1-x)*\ln(1-x)=[0*(-\infty)]=\left[\lim_{x\to 1}\frac{\ln(1-x)}{(1-x)^{-1}}\right]^H=\lim_{x\to 0}\frac{\frac{1}{1-x}*(-1)}{-1*(1-x)^{-2}*(-1)}=$$

$$=\lim_{x\to 1}\frac{-(1-x)^{\cancel{2}}}{\cancel{1-x}}=\mathbf{0}$$

E)

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right)=[\infty-\infty]=\lim_{x\to 0}\frac{\sin x-x}{xsinx}=\left[\frac{0}{0}\right]^H=\lim_{x\to 0}\frac{\cos x-1}{\sin x+xcosx}=\left[\frac{0}{0}\right]^H=$$

$$=\lim_{x\to 0}\frac{-\sin x}{\cos x+\cos x-xsinx}=\mathbf{0}$$

F)

$$\lim_{x\to 0^+}\left(\frac{1}{x}\right)^{\sin x}=[\infty^0]=\lim_{x\to 0^+}e^{\ln\left(\frac{1}{x}\right)^{\sin x}}=\lim_{x\to 0^+}e^{\sin x*\ln\left(\frac{1}{x}\right)}=e^{\lim\limits_{x\to 0^+}\sin x*\ln\left(\frac{1}{x}\right)}=$$

$$=\left[\lim_{x\to 0^+}\frac{\ln\left(\frac{1}{x}\right)}{(\sin x)^{-1}}\right]^H=\lim_{x\to 0^+}\frac{x*\frac{-1}{x^2}}{-1*(sinx)^{-2}*\cos x}=\left[\lim_{x\to 0}\frac{\sin^2 x}{xcosx}\right]^H=$$

$$=\lim_{x\to 0^+}\frac{2sinx*cosx}{cosx+x*(-sinx)}=\mathbf{0}$$

$$z=f(x,y)$$

$$f_x'=\frac{\delta f}{\delta x}\quad;\quad f_y'=\frac{\delta f}{\delta x}\quad;\quad f_{xx}'=\frac{\delta^2 f}{\delta^2 x}\quad;\quad f_{xy}''=\frac{\delta^2 f}{\delta y\delta x}\quad;\quad f''_{yx}=\frac{\delta^2 f}{\delta x\delta y}\quad;\quad f''_{yy}=\frac{\delta^2 f}{\delta y\delta y}$$

Przykład:

A)

$f(x,y) = 2x^2y - 3x + 2y - 5\sin(x,y)$

$f_x' = 2y * 2x - 3 + 0 - 5\cos(xy) * y = 4xy - 3 - 5y * \cos(xy)$

$f_y' = 2x^2 * 1 - 0 + 2 - 5 * \cos(xy) * x = 2x^2 + 2 - 5x * \cos(xy)$

$f"_{xx} = 4y * 1 - 5y * (-\sin(xy) * y)$

$f"_{xy} = 4x - 5(1 * \cos(xy) + y * (-\sin(xy) * x)$

$f"_{yx} = 4x - 5(1 * \cos(xy) + x * (-\sin(xy) * y)$

$f"_{yy} = -5x * (-\sin(xy) * x)$

B)

$f(x,y) = \frac{x}{y} = x * y^{-1}$

$f_x' = \frac{1}{y}$

$f_y' = x * (-y)^{-2}$

$f"_{xx} = 0$

$f"_{xy} = (y^{-1})' = -y^{-2}$

$f"_{yx} = -y^{-2}$

$f"_{yy} = x * 2y^{-3}$

C)

$f(x,y) = e^{xy} * \cos x$

$f_x' = e^{xy} * ycosx + e^{xy}(-\sin x)$

$f_y' = e^{xy} * \mathrm{x}\cos x$

$f"_{xx} = y * (e^{xy} * cosx + e^{xy} * (-sinx)) + e\hat{\ }xy * y((-sinx) + e^{xy} * (-cosx))$

$f"_{xy} = \cos x\,(e^{xy} * x + e^{xy}) + xy * x(-sinx)$

$f"_{yx} = e^{xy} * y * (x * cosx) + e^{xy} * \left(cosx + x(-sinx)\right)$

$f"_{yy} = x * cosx * e^{xy} * x$